Vulnerable Option Pricing: The Dual Problem

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Abstract

This paper derives an analytical formula to price vanilla options considering the counter-party risk under the first passage model, a credit risk model which is flexible and capable of describing typical market situations. Similar pricing methodology can also be applied to solve the dual problem – pricing vulnerable barrier options under Merton's structural model. Furthermore, our formulae can be used to evaluate structured notes, such as reverse exchangeable bond and barrier reverse convertibles, under the counter-party risk considerations.

Keywords: vulnerable option, credit risk, structured notes, reverse exchangeable

1 Introduction

Option-pricing theory has been well developed – but only disregarding the credit risk of counter-party. Many financial institutions trade derivatives through the over-the-counter (OTC) market with other financial institutions or their corporate clients. Since there are no margin and daily settlement mechanisms in the OTC market, the holders of these contracts are vulnerable to counter-party credit risk (see Johnson and Stulz, 1987). Accordingly, these kind of options, called vulnerable options, should have lower values than ordinary options, and the pricing problems should be considered.

When assessing the credit risk of a firm, the structural models are typically used. Merton's structural model (see Merton, 1974) assumes that a firm defaults if and only if its asset value can not meet the debt obligation at maturity date T. That is, defaults never occur before T. Empirical study suggests that this model is not able to describe the real world phenomenon (see Jarrow et al., 2003). Considering coupon payments and debt acceleration, the first passage time model is proposed by Black and Cox (1976) to improve Merton's model. In the first passage time model, a firm is assumed to default if its asset value is under a specified level. Therefore, this modified model allows premature default and is more flexible than Merton's model.

The pricing of vulnerable options has been discussed by existing literature. A comprehensive discussion is given by Johnson and Stulz (1987), who study how options subject to default risk under the assumptions that the value of the option writer's assets and the underlying asset is independent, and the option writer have no other liabilities which rank equally with payments under the option. Hull and White (1995) extend the Johnson and Stulz model such that other equal ranking claims on writer's assets can exist, but in their works the underlying and option writer's assets are still independent. This limitation is not released until Klein (1996), who obtain analytical pricing formulae which allows for correlation between the credit risk of the counter-party and the underlying asset. In the above works, however, the credit risk are all governed by Merton's model. This paper studies the methodology to evaluate vulnerable options, where the credit risk of counter-party is considered under a more flexible model – the first passage time model. Furthermore, in this methodology all of the above restrictions are released as well, and finally we obtain analytical pricing formulae.

Our methodology can also be generalized to solve the dual problem – pricing vulnerable barrier options under Merton's model. A barrier option is a popular exotic option whose payoff depends on whether the path of the underlying stock has reached a certain predetermined price level called barrier. Ignoring the credit risk, the analytical pricing formula for the barrier option can be derived (see Rubinstein and Reiner (1991)). As for the pricing problem of vulnerable barrier option under Merton's model, the case is a little more complex. Two processes with their own barriers should be considered: does the underlying asset value process ever hit the knock out barrier? and the other, is the counter-party's asset value lower than its debts at time T? The problem dealing with two processes and two barriers also appears in pricing a vulnerable vanilla option under the first passage model. What is involved are the underlying asset value process, concerning if it is in-the-money or out-of-the-money at T, and the counter-party's asset value process with default barrier during the

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life of this option. The above two pricing problems are dual, and the pricing formulae can be derived by similar methodologies. In this paper we will show how to evaluate the price of the down-and-out call option by the methodology we used to evaluate the vulnerable vanilla options, and then obtain the values of other kinds of barrier option through in-out parity.

Besides, our pricing methodology can be used to evaluate vulnerable structured notes, such as reverse exchangeable bonds and barrier reverse convertible bonds. A structured note is a debt instrument whose coupon payments or principal depends on the value of some underlying asset. Structured notes can also be thought of as a hybrid security that consist of a bond and a derivative. Since the tax law provides special discount rules that apply to these securities, the trading volume in structured notes keeps growing rapidly. When pricing a structured note, the issuer's credit risk should be considered as well. The Lehman Brothers' mini-bonds is an example that the issuer of structured notes fail to repay the coupon, even principal. As the risk does exist, the pricing problems for vulnerable structured notes should be taken seriously. This paper provides a pricing methodology to do so.

This paper is organized as follows. Section 2 gives the preliminaries which will be used in the following sections. Section 3, 4 gives the derivations of vulnerable vanilla and barrier options, respectively. Numerical examples are given in Section 5, and Section 6 conclude this paper.

2 Preliminaries

Consider a vanilla call option with initial time 0, maturity date T and strike price K, whose writer has firm value process V_t modeled by

$$\frac{dV_t}{V_t} = rdt + \sigma_V dZ_V(t), \tag{1}$$

and the underlying asset of this option has price process S_t following

$$\frac{dS_t}{S_t} = rdt + \sigma_S dZ_S(t), \tag{2}$$

where r is the riskless interest rate, σ_S, σ_V denote the volatility of the underlying and the writer, respectively, and $Z_S(t)$, $Z_V(t)$ are correlated standard Browinan motions with correlation given by

$$dZ_S(t)dZ_V(t) = \rho dt. \tag{3}$$

The main difference between Klein's and our work in pricing vulnerable vanilla option is in credit risk model. Their credit risk of counter-party is governed by Merton's model, whereas we consider the first passage time model.

In Merton's model (Merton, 1974), a firm is in default if its assets value is lower than promised repayment at maturity date T. Specifically, assuming the option writer has debt D to repay at T, the event of default occurs only if $V_T < D$. This assumption, however, is not very appropriate, since the information about the asset value of this firm during $t \in [0, t)$ is ignored. If the assets of the firm can be sold to meet the interest payments, a more realistic model is the first passage time model suggested by Black and Cox (1976), which assumes that there is a specified bankruptcy level $\tilde{L}(t)$, and the firm is in default whenever $V_t < \tilde{L}(t)$. Thus, a default can occur not only at time T but any time $t \in [0, T]$.

2.1 Vulnerable Vanilla Option Pricing under Merton's Structure Model

Derived by Klein (1996), a vulnerable vanilla option under Merton's model can be evaluated by an analytical pricing formula. In their work, if an option writer default, this option has recovery rate assumed to be δ, that is, the payoff of this option at T is $\delta\,(S_T - K)^+$ if writer's default occurs, and otherwise $(S_T - K)^+$, where $(\cdot)^+$ denotes $\max(\cdot, 0)$. In Merton's model, the counter-party defaults only if $V_T < D$, and hence the call value equals the discounted expected value of the payoff function

$$e^{-rT}\tilde{E}\left[(S_T - K)^+ 1_{\{V_T > D\}} + \delta\,(S_T - K)^+ 1_{\{V_T < D\}}\right] \tag{4}$$

where $1_{\{\cdot\}}$ denotes the indicator function, and $\tilde{E}$ is the expectation under risk neutral measure. Since the joint density function of (V_T, S_T) is analytically available, the closed-form formula for the above expectation can be derived.

It is also mentioned by Klein (1996) that the recovery rate δ can be approximated by $(1-\alpha)V_T/D$ under the premise that the payoff $(S_T - K)^+$ is much smaller than D, where α represents the bankruptcy cost. Indeed, if the counter-party defaults at time T, $(1-\alpha)V_T$ is received by liquidator, and the liability is $D+(S_T-K)^+$, where the option holder would ask for repayments

$$(1-\alpha)V_T\frac{(S_T-K)^+}{D+(S_T-K)^+} \quad\approx\quad (1-\alpha)V_T\frac{(S_T-K)^+}{D}.$$

Comparing this approximation with $\delta(S_T-K)^+$, it turns out that

$$\delta \approx (1-\alpha)V_T/D \tag{5}$$

We will follow this result later.

Klein (1996) didn't show how to extend their work under the first passage model. In fact, the above pricing methodology would fail when Merton's model is replaced by the first passage model. When the event of writer default is modeled by $V_t < L(t)$ for some $t \in [0,T]$, it becomes path-dependent. Since buyer's payoff is determined not only with the information of V_T, S_T but also V_t for all $t \in [0,T]$, only the density function of (V_T, S_T) is not enough to evaluate the call value. A more general pricing result is obtained in our paper, in which the first passage model is used, instead of Merton's model, and the pricing methodology leads to an analytical formula as well.

3 Pricing Vulnerable Options Under The First Passage Model

Assume that a vanilla call option with initial time 0, maturity date T and strike price K has writer's asset value process V_t modeled by Eq. (1) and underlying asset by Eq. (2). In this section the option writer's credit risk is considered under the first passage model, and the default boundary $\tilde{L}(t)$ is given by

$$\tilde{L}(t) \equiv \begin{cases} Le^{-\gamma(T-t)} & \forall t \in [0,T) \\ D & t = T \end{cases}, \tag{6}$$

where γ and L are predetermined constants and $L < D$. The writer's default occur if its firm value process V_t is less than $\tilde{L}(t)$ at some time $t \in [0,T]$. Let $\delta < 1$ be the recovery rate, that is, the option buyer will receives δ times of payoff if the counter-party default. Under this setting, the option buyer's payoff will vary in three cases:

- E_1: the writer never default during $[0,T]$ (ie. $V_t > \tilde{L}(t) \forall t \in [0,T) \cap V_T > D$)
- E_2: the writer never default before maturity date T but default at T (ie. $V_t > \tilde{L}(t) \forall t \in [0,T) \cap V_T < D$)
- E_3: the writer default before T (ie. $V_t < \tilde{L}(t)$ for some $t \in [0,T)$)

If event E_1 occurs, the buyer receives payoff $(S_T-K)^+$ at time T; if event E_2 occurs, the buyer receives payoff $\delta(S_T-K)^+$ at time T, and if E_3 occurs, the payoff becomes $\left(\delta e^{-r(T-\tau)}(S_T-K)^+\right)$ at time τ, where τ denotes the first hitting time $\tau \equiv \inf\{t \in [0,T] \mid V_t < \tilde{L}(t)\}$. Summing up and taking discounted expectation, we obtain the call value as follows

$$\begin{aligned} C = \quad & \tilde{E}\left[e^{-rT}(S_T-K)^+ 1_{E_1}\right] \\ + \quad & \delta\tilde{E}\left[e^{-rT}(S_T-K)^+ 1_{E_2}\right] \\ + \quad & \delta\tilde{E}\left[e^{-r\tau}\tilde{E}\left[e^{-r(T-\tau)}(S_T-K)^+\middle|\mathcal{F}_\tau\right]1_{E_3}\right]. \end{aligned}$$

To simplify the above expectation explicitly, notice that $\{E_1, E_2, E_3\}$ form a partition for all possible events. All of $E_1 \cap E_2$, $E_2 \cap E_3$, and $E_3 \cap E_1$ are empty sets, and all possible events are in $E_1 \cup E_2 \cup E_3$. Hence, the following identity holds:

$$\begin{aligned} C = \quad & \delta e^{-rT}\tilde{E}\left[(S_T-K)^+ 1_{E_1}\right] + (1-\delta)e^{-rT}\tilde{E}\left[(S_T-K)^+ 1_{E_1}\right] \\ + \quad & \delta e^{-rT}\tilde{E}\left[(S_T-K)^+ 1_{E_2}\right] \\ + \quad & \delta e^{-rT}\tilde{E}\left[(S_T-K)^+ 1_{E_3}\right] \\ = \quad & \delta e^{-rT}\tilde{E}\left[(S(T)-K)^+\right] \qquad (7) \\ + \quad & (1-\delta)e^{-rT}\tilde{E}\left[(S_T-K)1_{\{S_T>K\ \cap\ E_1\}}\right]. \qquad (8) \end{aligned}$$

Here we only need to focus on Eq. (8), since Eq. (7) can be evaluated by the Black-Scholes formula as

$$\delta\left(S_0 N_1(d_1) - Ke^{-rT}N_1(d_2)\right),$$

where

$$d_1 \equiv \frac{\log \frac{S_0}{K} + \left(r + \sigma_S^2/2\right) T}{\sigma_S \sqrt{T}}, \quad d_2 \equiv d_1 - \sigma_S \sqrt{T},$$

and $N_1(\cdot)$ is the cumulative distribution function of the standard normal distribution defined by

$$N_1(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Furthermore, applying Ito's lemma we have

$$\begin{cases} S_t = S_0 e^{\left(r - \sigma_S^2/2\right)t + \sigma_S Z_S(t)} \\ V_t = V_0 e^{\left(r - \sigma_V^2/2\right)t + \sigma_V Z_V(t)} \end{cases}.$$

Thus we can rewrite $S_T > K$ as

$$S_T = S_0 e^{\sigma_S B_2(T)} > K \Rightarrow B_2(T) \geq k, \tag{9}$$

where, for simplicity, we introduce a diffusion process $B_2(t)$ and a constant k defined by

$$\begin{aligned} B_2(t) &\equiv \frac{1}{\sigma_S}(r - \sigma_S^2/2)t + Z_S(t), \\ k &\equiv \frac{1}{\sigma_S} \log \frac{K}{S_0}. \end{aligned}$$

Similarly, $E_1 = (V_t > \tilde{L}(t) \forall t \in [0, T) \cap V_T > D)$ can be split and the event $V_T > D$ can be rewritten as

$$V_T = V_0 e^{\sigma_V B_1(T) + \gamma T} > D \Rightarrow B_1(T) > d,$$

where

$$\begin{aligned} B_1(t) &\equiv \frac{1}{\sigma_V}(r - \gamma - \sigma_V^2/2)t + Z_V(t), \\ d &\equiv \frac{1}{\sigma_V}\left(\log \frac{D}{V_0} - \gamma T\right), \end{aligned}$$

and the event $V_t > \tilde{L}(t) \forall t \in [0, T)$ can be rewritten as

$$\begin{aligned} & V_0 e^{\left(r - \sigma_V^2/2\right)t + \sigma_V Z_V(t)} = V_0 e^{\sigma_V B_1(t) + \gamma t} > L e^{-\gamma(T-t)} \quad \forall t \in [0, T) \\ \Rightarrow \quad & B_1(t) > l \quad \forall t \in [0, T) \\ \Rightarrow \quad & M_1(T) > l, \end{aligned}$$

where

$$\begin{aligned} M_1(t) &\equiv \min_{s \in [0,t]} B_1(s), \\ l &\equiv \frac{1}{\sigma_V}\left(\log \frac{L}{V_0} - \gamma T\right). \end{aligned}$$

Thus, Eq. (8) can be rewritten as

$$\begin{aligned} & (1-\delta)e^{-rT} \tilde{E}\left[(S_0 e^{\sigma_S B_2(T)} - K) 1_{\{B_2(T) > k, M_1(T) > l, B_1(T) > d\}}\right] \\ = \ & (1-\delta)e^{-rT} \int_k^{\infty} \int_d^{\infty} \int_l^{\infty} (S_0 e^{\sigma_S b_2} - K) f_{M_1(T), B_1(T), B_2(T)}(m_1, b_1, b_2) dm_1 db_1 db_2 \\ = \ & (1-\delta) S_0 e^{-rT} \int_k^{\infty} \int_d^{\infty} \int_l^{\infty} e^{\sigma_S b_2} f_{M_1(T), B_1(T), B_2(T)}(m_1, b_1, b_2) dm_1 db_1 db_2 \\ & -(1-\delta) K e^{-rT} \int_k^{\infty} \int_d^{\infty} \int_l^{\infty} f_{M_1(T), B_1(T), B_2(T)}(m_1, b_1, b_2) dm_1 db_1 db_2, \end{aligned} \tag{10}$$

where $f_{M_1(T), B_1(T), B_2(T)}$ denotes the joint probability density function of $M_1(T), B_1(T), B_2(T)$, and $B_1(t), B_2(t)$ are Brownian motions with drift terms with correlation $dB_1(t) dB_2(t) = \rho dt$. [1] As $f_{M_1(T), B_1(T), B_2(T)}$ can be derived explicitly (see Appendix A), the above integrals can be carried out analytically. To do this, it is convenient to introduce the following universal formula first:

[1] $dB_1(t) dB_2(t) = \left(\frac{1}{\sigma_V}(r - \gamma - \sigma_V^2/2)dt + dZ_V(t)\right)\left(\frac{1}{\sigma_S}(r - \sigma_S^2/2)dt + dZ_S(t)\right) = dZ_V(t) dZ_S(t) = \rho dt$ by Eq. (3).

Theorem 3.1 *Assume $B_1(t), B_2(t)$ are correlated Brownian motions with drift terms αt, βt, and $dB_1(t)dB_2(t) = \rho dt$. Then*

$$\begin{aligned}
&\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty} e^{pb_1+qb_2} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2 \\
= \;& e^{T\left(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q\right)} N_2\left(-\frac{\bar{b}_1}{\sqrt{T}}+\sqrt{T}(\alpha+p+\rho q), -\frac{\bar{b}_2}{\sqrt{T}}+\sqrt{T}(\beta+\rho p+q),\rho\right) \\
&-e^{T\left(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q\right)+2\bar{m}_1(p+\rho q)+2\alpha\bar{m}_1} N_2\left(-\frac{\bar{b}_1-2\bar{m}_1}{\sqrt{T}}+\sqrt{T}\left(\alpha+p+\rho q\right), -\frac{\bar{b}_2-2\rho\bar{m}_1}{\sqrt{T}}+\sqrt{T}\left(\beta+\rho p+q\right),\rho\right),
\end{aligned}$$

where $N_2(x, y, \rho)$ is the cumulative distribution function of a bivariate standard normal distribution defined by

$$N_2(x,y,\rho) \equiv \int_{-\infty}^{y}\int_{-\infty}^{x} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(u^2-2\rho uv+v^2\right)} du dv.$$

Proof. See Appendix B. □

With Theorem 3.1, Eq. (10) can be simplified to obtain the following pricing formula:

$$\begin{aligned}
C \;=\;& (1-\delta)S_0\Bigg[N_2\left(-\frac{d}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_V}(r-\gamma-\sigma_V^2/2+\rho\sigma_S\sigma_V), -\frac{k}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_S}(r+\sigma_S^2/2),\rho\right) \\
&-e^{2\rho\sigma_S l+\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_2\left(\frac{2l-d}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_V}(r-\gamma-\sigma_V^2/2+\rho\sigma_S\sigma_V), \frac{2\rho l-k}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_S}(r+\sigma_S^2/2),\rho\right)\Bigg] \\
&-(1-\delta)Ke^{-rT}\Bigg[N_2\left(-\frac{d}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_V}(r-\gamma-\sigma_V^2/2), -\frac{k}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_S}(r-\sigma_S^2/2),\rho\right) \\
&-e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_2\left(\frac{2l-d}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_V}(r-\gamma-\sigma_V^2/2), \frac{2\rho l-k}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_S}(r-\sigma_S^2/2),\rho\right)\Bigg] \\
&+\delta\left(S_0N_1(d_1)-Ke^{-rT}N_1(d_2)\right).
\end{aligned} \tag{11}$$

Recall that

$$d_1 = \frac{\log\frac{S_0}{K}+\left(r+\sigma_S^2/2\right)T}{\sigma_S\sqrt{T}}, \qquad d_2 = \frac{\log\frac{S_0}{K}+\left(r-\sigma_S^2/2\right)T}{\sigma_S\sqrt{T}}.$$

Setting

$$d_3 \equiv \frac{\log\frac{V_0}{D}+\left(r+\sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}, \qquad d_4 \equiv \frac{\log\frac{V_0}{D}+\left(r-\sigma_V^2/2\right)T}{\sigma_V\sqrt{T}},$$

the above mess can be further simplified. Since direct computations show that $-\frac{k}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_S}(r+\sigma_S^2/2)$ in Eq. (11) equals d_1, $-\frac{d}{\sqrt{T}}+\frac{\sqrt{T}}{\sigma_V}(r-\gamma-\sigma_V^2/2) = d_4$, and hence the whole Eq. (11) can be rewritten as $(1-\delta)S_0N_2\left(d_4+\sqrt{T}\rho\sigma_S, d_1,\rho\right)$. Similarly, other terms can also be further simplified, and it finally turns out the pricing formula for vulnerable vanilla call

$$\begin{aligned}
C \;=\;& (1-\delta)S_0\left[N_2\left(d_4+\sqrt{T}\rho\sigma_S, d_1,\rho\right) - \underbrace{e^{2\rho\sigma_S l+\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_2\left(d_4+\frac{2l}{\sqrt{T}}+\sqrt{T}\rho\sigma_S, d_1+\frac{2\rho l}{\sqrt{T}},\rho\right)}_{\mathbf{A}}\right] \\
&-(1-\delta)Ke^{-rT}\left[N_2\left(d_4,d_2,\rho\right) - \underbrace{e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_2\left(d_4+\frac{2l}{\sqrt{T}}, d_2+\frac{2\rho l}{\sqrt{T}},\rho\right)}_{\mathbf{B}}\right] + \delta\left(S_0N_1(d_1)-Ke^{-rT}N_1(d_2)\right).
\end{aligned} \tag{12}$$

A vulnerable vanilla put option can also be evaluated by the aforementioned methodology. The analytical pricing formula is given by

$$\begin{aligned} P &= -(1-\delta)S_0\left[N_2\left(d_4+\sqrt{T}\rho\sigma_S,-d_1,-\rho\right)-e^{2\rho\sigma_S l+\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)}N_2\left(d_4+\frac{2l}{\sqrt{T}}+\sqrt{T}\rho\sigma_S,-d_1-\frac{2\rho l}{\sqrt{T}},-\rho\right)\right] \\ &\quad +(1-\delta)Ke^{-rT}\left[N_2\left(d_4,-d_2,-\rho\right)-e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)}N_2\left(d_4+\frac{2l}{\sqrt{T}},-d_2-\frac{2\rho l}{\sqrt{T}},-\rho\right)\right] \\ &\quad +\delta\left(-S_0N_1(-d_1)+Ke^{-rT}N_1(-d_2)\right). \end{aligned} \tag{13}$$

3.1 Reduce to Black-Scholes Formula

The value of a vulnerable vanilla option given in Eq. (12) really boils down to the Black-Scholes formula when the credit risk of writer is ignored. To illustrate this, note that the Black-Scholes pricing formula for a vanilla call option is given by

$$C_{BS}=S_0N_1(d_1)-Ke^{-rT}N_1(d_2).$$

Since the last term of Eq. (12) is just δ times of C_{BS}, we only need to show the rest terms reduces to $(1-\delta)C_{BS}$ to verify the whole reduction. Setting default boundary $\tilde{L}(t)$ to zero, we have $D\to 0$, $L\to 0$, and hence

$$\ell=\left(\log\frac{L}{V_0}-\gamma T\right)/\sigma_V\to-\infty,\qquad d_4=\frac{\log\frac{V_0}{D}+\left(r-\sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}\to\infty.$$

Thus, $-e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)}$ in block **B** of Eq. (12) tends to zero. Since $N_2(\cdot)\le 1$ by its definition, the whole block **B** vanishes. A similar argument shows that block **A** also vanishes. On the other hand, as d_4 tends to infinity, $N_2(d_4+\sqrt{T}\rho\sigma_S,d_1,\rho)$ and $N_2(d_4,d_2,\rho)$ would reduce to $N_1(d_1)$ and $N_1(d_2)$, respectively.[2]Summing up the above observations leads us to the conclusion. Similarly, Eq. (13) reduces to the Black-Scholes pricing formula for a put option if the credit risk consideration is taken away (ie. $D\to 0$, $L\to 0$).

3.2 Extension to Structured Note Pricing

The aforementioned pricing methodology can also be used to evaluate the price of a vulnerable structured note. As an example, below we will show how to price a reverse exchangeable bound under the first passage model. A reverse exchangeable bond is a structured note, in which the holder will receive returns depending on an underlying asset at maturity date T. Specifically, let S_t be the value process of underlying asset. The holder receives principal and coupon if $S_T\ge S_0$. Otherwise, if $S_T<S_0$, the holder receives principal and underlying. A reverse exchangeable bound can be thought of as a hybrid security that contains a straight bond tied together with a short position of put option with strike $K=S_0$. Hence, to price a vulnerable structured note, we only need to evaluate a vulnerable straight bond subtracted by a vulnerable vanilla put option. Since a vulnerable vanilla put has pricing formula given by Eq. (13), now we only focus on vulnerable bond pricing. Let a bond has face value F and coupon rate c, and its issuer has firm value process modeled by Eq. (1). Now the issuer's credit risk is also modeled by the first passage model with default boundary Eq. (6). The bond holder's payoff varies in these cases:

- E_1 occurs and holder's payoff at T is $(F+FcT)$
- E_2 occurs and holder's payoff at T is $\delta(F+FcT)$
- E_3 occurs and holder's payoff is $\delta e^{-r(T-\tau)}(F+FcT)$ at time τ

Here δ denotes the recovery rate of the issuer of this bond, and τ is the first hitting time that $V_t<\tilde{L}(t)$. Taking discounted expectation under risk neutral measure, we have the vulnerable bond price

$$\begin{aligned} VB &= \tilde{E}\left[e^{-rT}(F+FcT)1_{E_1}\right]+\delta\tilde{E}\left[e^{-rT}(F+FcT)1_{E_2}\right]+\delta\tilde{E}\left[e^{-r\tau}\tilde{E}\left[e^{-r(T-\tau)}(F+FcT)\middle|\mathcal{F}_\tau\right]1_{E_3}\right] \\ &= \underbrace{\tilde{E}\left[e^{-rT}(F+FcT)1_{E_1}\right]}_{\mathbf{C}}+\delta\tilde{E}\left[e^{-rT}(F+FcT)1_{E_2}\right]+\delta\tilde{E}\left[e^{-rT}(F+FcT)1_{E_3}\right]. \end{aligned}$$

Again, we rewrite block **C** as $(\delta\mathbf{C}+(1-\delta)\mathbf{C})$ to obtain

$$\begin{aligned} VB &= e^{-rT}\delta(F+FcT)+e^{-rT}(1-\delta)(F+FcT)\tilde{E}\left[1_{E_1}\right] \\ &= e^{-rT}(F+FcT)\left(\delta+(1-\delta)\tilde{E}\left[1_{E_1}\right]\right). \end{aligned} \tag{14}$$

[2]Notice that $\lim_{x\to\infty}N_2(x,y,\rho)=N_1(y)$ for all ρ,y. A brief derivation is given in Appendix C.

Recall that $E_1 = (B_1(T) > d \cap M_1(T) > l)$, and hence $\tilde{E}[1_{E_1}]$ can be rewritten as

$$
\begin{aligned}
\tilde{E}[1_{E_1}] &= \tilde{E}\left[1_{B_1(T)>d, M_1(T)>l}\right] \\
&= \int_{-\infty}^{\infty}\int_{d}^{\infty}\int_{l}^{\infty} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1 db_1 db_2 \qquad (15)\\
&= N_1\left(-\frac{d}{\sqrt{T}} + \frac{\sqrt{T}}{\sigma_V}(r-\gamma-\sigma_V^2/2)\right) - e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_1\left(\frac{2l-d}{\sqrt{T}} + \frac{\sqrt{T}}{\sigma_V}\left(r-\gamma-\sigma_V^2/2\right)\right) \\
&= N_1(d_4) - e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_1\left(d_4 + \frac{2l}{\sqrt{T}}\right), \qquad (16)
\end{aligned}
$$

where Theorem 3.1 is used again to evaluate Eq. (15). Substituting Eq. (16) into Eq. (14), we obtain the vulnerable straight bond value

$$
VB = e^{-rT}(F+FcT)\left(\delta + (1-\delta)\left(N_1(d_4) - e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_1\left(d_4 + \frac{2l}{\sqrt{T}}\right)\right)\right),
$$

and, consequently, the pricing formula for a vulnerable reverse exchangeable bond can be derived as

$$
\begin{aligned}
& VB - \left(\frac{F}{K}\right)P = \\
& e^{-rT}(F+FcT)\left(\delta + (1-\delta)\left(N_1(d_4) - e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_1\left(d_4 + \frac{2l}{\sqrt{T}}\right)\right)\right) \\
-\ & \left(\frac{F}{K}\right)\left(-(1-\delta)S_0\left[N_2\left(d_4 + \sqrt{T}\rho\sigma_S, -d_1, -\rho\right) - e^{2\rho\sigma_S l + \frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_2\left(d_4 + \frac{2l}{\sqrt{T}} + \sqrt{T}\rho\sigma_S, -d_1 - \frac{2\rho l}{\sqrt{T}}, -\rho\right)\right]\right. \\
+\ & (1-\delta)Ke^{-rT}\left[N_2(d_4, -d_2, -\rho) - e^{\frac{2l}{\sigma_V}(r-\gamma-\sigma_V^2/2)} N_2\left(d_4 + \frac{2l}{\sqrt{T}}, -d_2 - \frac{2\rho l}{\sqrt{T}}, -\rho\right)\right] \\
+\ & \left.\delta\left(-S_0 N_1(-d_1) + Ke^{-rT}N_1(-d_2)\right)\right).
\end{aligned}
$$

4 Pricing Vulnerable Barrier Options Under Merton's Model

Now we turn to evaluate the value of a vulnerable barrier options under Merton's model. Among all types of barrier options, we only give the detail of the derivation of the vulnerable down-and-out call value. Pricing formulae for other types of barrier options are given in appendix E, and their derivations are similar.

Consider a vulnerable down-and-out call option with initial time 0, maturity date T and strike price K, whose writer and underlying asset are also modeled by the processes Eqs (1), (2). Here writer's default is governed by Merton's model; default only occurs when $V_T < D$, and a lower barrier for underlying asset is introduced, which is defined by $\tilde{B}(t) \equiv Be^{-\bar{\gamma}(T-t)}, \quad \forall t \in [0,T]$.

During the life of this option, once the underlying asset value hits the lower barrier, a option buyer gets no return either the writer defaults or not. On the other hand, if the underlying asset value never cross the lower barrier in $[0,T]$, buyer's payoff depends on whether writer defaults or not. The option buyer receives payoff $\delta(S_T - K)^+$ if a default occurs, and otherwise $\delta(S_T-K)^+$. Thus, the value of a down-and-out call is given by the discounted expectation as follows

$$
\begin{aligned}
C_{DO} &= \tilde{E}\left[e^{-rT}(S_T-K)^+ 1_{\{S_t>\tilde{B}(t)\forall t\in[0,T]\ \cap\ V_T>D\}}\right] + \tilde{E}\left[e^{-rT}\delta(S_T-K)^+ 1_{\{S_t>\tilde{B}(t)\forall t\in[0,T]\ \cap\ V_T<D\}}\right] \qquad (17)\\
&= \tilde{E}\left[e^{-rT}(S_T-K)^+ 1_{\{S_t>\tilde{B}(t)\forall t\in[0,T]\ \cap\ V_T>D\}}\right] \\
&\quad + \tilde{E}\left[e^{-rT}\frac{(1-\alpha)V_T}{D}(S_T-K)^+ 1_{\{S_t>\tilde{B}(t)\forall t\in[0,T]\ \cap\ V_T<D\}}\right] \qquad (18)\\
&= e^{-rT}\tilde{E}\left[(S_T-K)1_{\{S_T>K\ \cap\ S_t>\tilde{B}(t)\forall t\in[0,T]\ \cap\ V_T>D\}}\right] \\
&\quad + e^{-rT}\tilde{E}\left[\frac{(1-\alpha)V_T}{D}(S_T-K)1_{\{S_T>K\ \cap\ S_t>\tilde{B}(t)\forall t\in[0,T]\ \cap\ V_T<D\}}\right], \qquad (19)
\end{aligned}
$$

where the approximation for recovery rate given in Eq. (5) is substituted into Eq. (17) to obtain Eq. (18).

To simplify the above expectation, we set

$$
B_1(t) \equiv \tfrac{1}{\sigma_S}(r-\bar{\gamma}-\sigma_S^2/2)t + Z_S(t), \quad M_1(t) \equiv \min_{s\in[0,t]} B_1(s), \quad B_2(t) \equiv \tfrac{1}{\sigma_V}(r-\sigma_V^2/2)t + Z_V(t),
$$

$$
k' \equiv \tfrac{1}{\sigma_S}\left(\log\tfrac{K}{S_0} - \bar{\gamma}T\right), \qquad b \equiv \tfrac{1}{\sigma_S}\left(\log\tfrac{B}{S_0} - \bar{\gamma}T\right), \qquad d' \equiv \tfrac{1}{\sigma_V}\log\tfrac{D}{V_0}.
$$

Thus, similarly to Eq. (9), the event $S_T > K$ can be reduced as follows:

$$\begin{aligned}
& S_T = S_0 e^{\left(r-\sigma_S^2/2\right)T+\sigma_S Z_S(T)} > K \\
\Rightarrow\quad & S_0 e^{\sigma_S\left(\frac{1}{\sigma_S}\left(r-\bar{\gamma}-\sigma_S^2/2\right)T+Z_S(T)\right)} = S_0 e^{\sigma_S B_1(T)} > K e^{-\bar{\gamma}T} \\
\Rightarrow\quad & B_1(T) > \frac{1}{\sigma_S}\left(\log\frac{K}{S_0}-\bar{\gamma}T\right) = k'.
\end{aligned}$$

Similarly, we have the reductions given in the following table:

The original form	After reduction	In simplified notions
$S_T > K$	$\frac{1}{\sigma_S}\left(r-\bar{\gamma}-\frac{\sigma_S^2}{2}\right)T+Z_S(T) > \frac{1}{\sigma_S}\left(\log\frac{K}{S_0}-\bar{\gamma}T\right)$	$B_1(T) > k'$
$S_t > \tilde{B}(t)\forall t\in[0,T]$	$\min_{t\in[0,T]}\left(\frac{1}{\sigma_S}\left(r-\bar{\gamma}-\frac{\sigma_S^2}{2}\right)t+Z_S(t)\right) > \frac{1}{\sigma_S}\left(\log\frac{B}{S_0}-\bar{\gamma}T\right)$	$M_1(T) > b$
$V_T > D$	$\frac{1}{\sigma_V}\left(r-\frac{\sigma_V^2}{2}\right)T+Z_V(T) > \frac{1}{\sigma_V}\log\frac{D}{V_0}$	$B_2(T) > d'$
$V_T < D$	$\frac{1}{\sigma_V}\left(r-\frac{\sigma_V^2}{2}\right)T+Z_V(T) < \frac{1}{\sigma_V}\log\frac{D}{V_0}$	$B_2(T) < d'$

Thus, Eq. (19) can be rewritten as

$$\begin{aligned}
C_{DO} &= e^{-rT}\tilde{E}\left[(S_0e^{\sigma_S B_1(T)+\bar{\gamma}T}-K)1_{\{B_1(T)>k'\ \cap\ M_1>b\ \cap\ B_2(T)>d'\}}\right] \\
&\quad+\left(\frac{1-\alpha}{D}\right)e^{-rT}\tilde{E}\left[V_0e^{\sigma_V B_2(T)}(S_0e^{\sigma_S B_1(T)+\bar{\gamma}T}-K)1_{\{B_1(T)>k'\ \cap\ M_1>b\ \cap\ B_2(T)<d'\}}\right] \\
&= S_0e^{(\bar{\gamma}-r)T}\int_{d'}^{\infty}\int_{k'}^{\infty}\int_{b}^{\infty}e^{\sigma_S b_1}f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2 && (20)\\
&\quad-Ke^{-rT}\int_{d'}^{\infty}\int_{k'}^{\infty}\int_{b}^{\infty}f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2 && (21)\\
&\quad+\left(\frac{1-\alpha}{D}\right)S_0V_0e^{(\bar{\gamma}-r)T}\int_{-\infty}^{d'}\int_{k'}^{\infty}\int_{b}^{\infty}e^{\sigma_S b_1+\sigma_V b_2}f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2 && (22)\\
&\quad-\left(\frac{1-\alpha}{D}\right)KV_0e^{-rT}\int_{-\infty}^{d'}\int_{k'}^{\infty}\int_{b}^{\infty}e^{\sigma_V b_2}f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2, && (23)
\end{aligned}$$

where Eqs. (20) and (21) can be simplified by Theorem 3.1. As for Eqs. (22) and (23), since the original formula becomes invalid due to the change of limits of integration, we introduce the following modification and left its proof in Appendix D:

Theorem 4.1 *Assume $B_1(t), B_2(t)$ are correlated Brownian motions with drift terms αt, βt, and $dB_1(t)dB_2(t) = \rho dt$. Then*

$$\begin{aligned}
&\int_{-\infty}^{\bar{b}_2}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty}e^{pb_1+qb_2}f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2 \\
=\ & e^{T\left(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q\right)}N_2\left(-\frac{\bar{b}_1}{\sqrt{T}}+\sqrt{T}(\alpha+p+\rho q),\frac{\bar{b}_2}{\sqrt{T}}-\sqrt{T}(\beta+\rho p+q),-\rho\right) \\
&-e^{T\left(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q\right)+2\bar{m}_1(p+\rho q)+2\alpha\bar{m}_1}N_2\left(-\frac{\bar{b}_1-2\bar{m}_1}{\sqrt{T}}+\sqrt{T}\left(\alpha+p+\rho q\right),\frac{\bar{b}_2-2\rho\bar{m}_1}{\sqrt{T}}-\sqrt{T}\left(\beta+\rho p+q\right),-\rho\right).
\end{aligned} \quad (24)$$

As a result, Eqs (20), (21), (22), (23) can be simplified to yield the pricing formula for a vulnerable down-and-out call option

$$
\begin{aligned}
&C_{DO} = \\
&S_0 \left[N_2\left(d_1, d_4 + \sqrt{T}\rho\sigma_S, \rho\right) \underbrace{- e^{\frac{2b}{\sigma_S}(r-\bar{\gamma}+\sigma_S^2/2)} N_2\left(d_1 + \frac{2b}{\sqrt{T}}, d_4 + \frac{2\rho b}{\sqrt{T}} + \sqrt{T}\rho\sigma_S, \rho\right)}_{\mathbf{C}} \right] \\
&-Ke^{-rT} \left[N_2\left(d_2, d_4, \rho\right) \underbrace{- e^{\frac{2b}{\sigma_S}(r-\bar{\gamma}-\sigma_S^2/2)} N_2\left(d_2 + \frac{2b}{\sqrt{T}}, d_4 + \frac{2\rho b}{\sqrt{T}}, \rho\right)}_{\mathbf{D}} \right] \\
&+\left(\frac{1-\alpha}{D}\right) S_0 V_0 e^{(r+\rho\sigma_S\sigma_V)T} \left[\underbrace{N_2\left(d_1 + \sqrt{T}\rho\sigma_V, -d_3 - \sqrt{T}\rho\sigma_S, -\rho\right)}_{\mathbf{E}} \right. \\
&\left. \underbrace{- e^{\frac{2b}{\sigma_S}(r-\bar{\gamma}+\sigma_S^2/2+\rho\sigma_S\sigma_V)} N_2\left(d_1 + \frac{2b}{\sqrt{T}} + \sqrt{T}\rho\sigma_V, -d_3 - \frac{2\rho b}{\sqrt{T}} - \sqrt{T}\rho\sigma_S, -\rho\right)}_{\mathbf{F}} \right] \\
&-\left(\frac{1-\alpha}{D}\right) KV_0 \underbrace{\left[N_2\left(d_2 + \sqrt{T}\rho\sigma_V, -d_3, -\rho\right) - e^{\frac{2b}{\sigma_S}(r-\bar{\gamma}-\sigma_S^2/2+\rho\sigma_S\sigma_V)} N_2\left(d_2 + \frac{2b}{\sqrt{T}} + \sqrt{T}\rho\sigma_V, -d_3 - \frac{2\rho b}{\sqrt{T}}, -\rho\right) \right]}_{\mathbf{G}}.
\end{aligned} \tag{25}
$$

There are 8 types of vulnerable barrier options: down-and-out call (denoted by C_{DO}), down-and-out put (P_{DO}), down-and-in call (C_{DI}), down-and-in put (P_{DI}), up-and-out call (C_{UO}), up-and-out put (P_{UO}), up-and-in call (C_{UI}), and up-and-in put (P_{UI}). Thought all of these can be priced by the above methodology, thanks to in-out parity, we only need to evaluated half of them. Indeed, in-out parity states that a vanilla call equals the sum of a down-and-out call and a down-and-in call, no matter if the credit risk of counter-party is considered or not. Hence, we have

$$C_{DO} + C_{DI} = \text{The value of a vulnerable vanilla call},$$

where the value of a vulnerable vanilla call under Merton's model has been derived by Klein (1996). We just use it and the pricing formula for C_{DO} given in Eq. (25) to determine the value of C_{DI}. Similarly, since the following identities hold:

$$
\begin{aligned}
C_{UO} + C_{UI} &= \text{The value of a vulnerable vanilla call}, \\
P_{DO} + P_{DI} &= P_{UO} + P_{UI} = \text{The value of a vulnerable vanilla put},
\end{aligned}
$$

and the value of a vulnerable vanilla put is also given by Klein (1996), we only need to evaluate P_{DI}, C_{UO}, and P_{UI} to obtain the value of all 8 types of vulnerable barrier options. The pricing formulae for P_{DI}, C_{UO}, and P_{UI} are given in Appendix E.

4.1 Reductions

There are several perspectives for the pricing formula Eq. (25). In some trading outside the OTC market, counter-party's credit risk is very small and hence can be neglected. In this non-vulnerable case, our pricing formula provides the flexibility to reduce to the barrier option value. On the other hand, a barrier option reduces to a vanilla one when the barrier is taken away. Our formulae are also able to degenerate the vulnerable vanilla option valuation.

4.1.1 Reduce to Non-vulnerable Case

Without credit risk considerations, the pricing formula for a down-and-out call has obtained by Rubinstein and Reiner (1991), which is given by

$$S_0\left[N_1\left(d_1\right) - \left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma_S^2}+1} N_1\left(d_1 + \frac{2\log\frac{B}{S_0}}{\sigma_S\sqrt{T}}\right)\right] - Ke^{-rT}\left[N_1\left(d_2\right) - \left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma_S^2}-1} N_1\left(d_2 + \frac{2\log\frac{B}{S_0}}{\sigma_S\sqrt{T}}\right)\right]. \tag{26}$$

This formula is a special case of Eq. (25). Indeed, when the counter-party's default barrier D tens to zero, we have $\log \frac{V_0}{D} \to \infty$ and hence $d_3, d_4 \to \infty$, and $-d_3, -d_4 \to -\infty$. Thus, block $\mathbf{E}, \mathbf{F}, \mathbf{G}$ in Eq. (25) vanish, and all of the other $N_2(\cdot)$ reduce to $N_1(\cdot)$. Now Eq. (25) becomes

$$S_0 \left[N_1 (d_1) - e^{\frac{2b}{\sigma_S}(r-\bar{\gamma}+\sigma_S^2/2)} N_1 \left(d_1 + \frac{2b}{\sqrt{T}} \right) \right] - Ke^{-rT} \left[N_1 (d_2) - e^{\frac{2b}{\sigma_S}(r-\bar{\gamma}-\sigma_S^2/2)} N_1 \left(d_2 + \frac{2b}{\sqrt{T}} \right) \right]. \tag{27}$$

Furthermore, in Rubinstein's work a constant barrier $\tilde{B}(t) = B$ is considered, and thus $\bar{\gamma} = 0$, $b = \frac{1}{\sigma_S} \left(\log \frac{B}{S_0} \right)$. Substituting this into Eq. (27) we obtain Eq. (26).

4.1.2 Reduce to Klein's Pricing Formula

Eq. (25) also reduces to Klein's pricing formula for vulnerable vanilla call under Merton's model (see Klein (1996)) as $B \to 0$. The argument goes as follows. As $B \to 0$ we have $b = \frac{1}{\sigma_S} \left(\log \frac{B}{S_0} - \bar{\gamma} T \right) \to -\infty$, and $e^{\frac{2b}{\sigma_S}} \to 0$. Thus, in Eq. (25), block $\mathbf{C}, \mathbf{D}, \mathbf{F}$ tends to zero, and so does the last term of block $\mathbf{G}$. As a result, Eq. (25) reduces to

$$\begin{aligned} & S_0 N_2 \left(d_1, d_4 + \sqrt{T}\rho\sigma_S, \rho \right) - Ke^{-rT} N_2 (d_2, d_4, \rho) \\ & + \left(\frac{1-\alpha}{D} \right) S_0 V_0 e^{(r+\rho\sigma_S\sigma_V)T} N_2 \left(d_1 + \sqrt{T}\rho\sigma_V, -d_3 - \sqrt{T}\rho\sigma_S, -\rho \right) - \left(\frac{1-\alpha}{D} \right) K V_0 N_2 \left(d_2 + \sqrt{T}\rho\sigma_V, -d_3, -\rho \right), \end{aligned}$$

which is just Klein's formula.

4.2 Extension to Structured Note Pricing

Our pricing scheme can be extended to evaluate more complex financial instruments. As an example, the derivation of the analytical pricing formula for a barrier reverse convertible security is given below.

A barrier reverse convertible security is a note linked to an underlying asset, where we assume the value of this underlying asset at time t is modeled by S_t. At maturity T, the holder will receive the coupon and underlying, if S_t ever hit a predetermined lower barrier $\tilde{B}(t)$ during $t \in [0, T]$ and S_T is less than the initial price S_0. Otherwise, if $S_T > S_0$ or $S_t > \tilde{B}(t) \forall t \in [0, T]$, the holder will receive the coupon and principal. The value of such security with counter-party's credit risk consideration can be evaluated by the above pricing methodology for vulnerable barrier options. A barrier reverse convertible can be thought of as a straight bond tied together with a short position of down-and-in put option with strike $K = S_0$. Thus, its price equals a vulnerable bond price subtracted by a down-and-in put option.

Under Merton's model, pricing a vulnerable bond is even more easy than first passage model case. Since default only occur when $V_T < D$, the bond holder's payoff varies in two cases:

- If $V_T > D$, holder's payoff at T is $(F + FcT)$
- If $V_T < D$, holder's payoff at T is $\delta(F + FcT) = \frac{(1-\alpha)V_T}{D}(F + FcT)$

Thus, we have the vulnerable bond price under Merton's model as

$$\begin{aligned} VF &= \tilde{E}\left[e^{-rT}(F + FcT)1_{\{V_T > D\}}\right] + \tilde{E}\left[e^{-rT}\frac{(1-\alpha)V_T}{D}(F + FcT)1_{\{V_T < D\}}\right] \\ &= e^{-rT}(F + FcT)\tilde{E}\left[1_{\{V_T > D\}}\right] + \frac{(1-\alpha)}{D}(F + FcT)e^{-rT}\tilde{E}\left[V_T 1_{\{V_T < D\}}\right], \end{aligned}$$

where $\tilde{E}\left[1_{\{V_T > D\}}\right]$ can be evaluated by Theorem 3.1 as follows

$$\begin{aligned} \tilde{E}\left[1_{\{V_T > D\}}\right] &= \tilde{E}\left[1_{\{B_1(T) > d\}}\right] \\ &= \int_{-\infty}^{\infty}\int_{d}^{\infty}\int_{-\infty}^{\infty} f_{M_1(T), B_1(T), B_2(T)}(m_1, b_1, b_2)dm_1 db_1 db_2 \\ &= N_1\left(-\frac{d}{\sqrt{T}} + \frac{\sqrt{T}}{\sigma_V}(r - \gamma - \sigma_V^2/2)\right) = N_1(d_4). \end{aligned} \tag{28}$$

As for $\tilde{E}\left[V_T 1_{\{V_T < D\}}\right]$, note that

$$\begin{aligned} \tilde{E}\left[V_T 1_{\{V_T < D\}}\right] &= D\tilde{E}\left[1_{\{V_T < D\}}\right] - \tilde{E}\left[(D - V_T)1_{\{V_T < D\}}\right] \\ &= D\left(1 - \tilde{E}\left[1_{\{V_T > D\}}\right]\right) - \tilde{E}\left[(D - V_T)^+\right]. \end{aligned}$$

Since the expectation in the first term of the right hand side has been evaluated in Eq. (28), and the second term can be evaluated by the Black-scholes formula for put options, we obtain

$$\begin{aligned}\tilde{E}\left[V_T 1_{\{V_T<D\}}\right] &= D(1-N_1(d_4)) - \tilde{E}\left[(D-V_T)^+\right] \\ &= D(1-N_1(d_4)) - \big(DN_1(-d_4) - V_0 e^{rT} N_1(-d_3)\big) \\ &= V_0 e^{rT} N_1(-d_3).\end{aligned}$$

Consequently, we have

$$VF \;=\; e^{-rT}(F+FcT)N_1(d_4) + \frac{(1-\alpha)}{D}(F+FcT)V_0 N_1(-d_3).$$

Subtracted by P_{DI}, finally we obtain the pricing formula for a vulnerable barrier reverse convertibles.

5 Numerical Results

As mentioned before, ignoring the default risk of counter party would cause great danger. In this section we show how significant a derivative could be mispriced if the default risk of counter party is ignored. With parameters $K=40, V(0)=100, T=3, D=90, L=70, \sigma_V=0.2, \rho=0, r=0.05, \gamma=0.06, \delta=0.6$, the values of vanilla calls are shown in panel (a) of Figure 1, where $S(0)$ varies from 20 to 60, the vulnerable call prices are evaluated by Eq. (12), and the non-vulnerable call price, the Black-Scholes formula. Since the payoff function $(S_T-K)^+$ has more probability to be large if $S(0)$ is large, the value of a vanilla call increases as $S(0)$ increases. This phenomenon can be observed in case of a vulnerable option, as well as a non-vulnerable one. It is also notable that as shown in panel (a) of Figure 1, the value of a vulnerable option is lower than a non-vulnerable one. The difference between these two derivatives due to the default risk of counter party can be thought of as a risk premium, which is even more significant when $S(0)$ is fixed and $V(0)$ is changed, as shown in panel (b) of Figure 1. Similarly, under the same setting but different σ_V, the values of a vulnerable and a non-vulnerable call also has difference varied significantly. As shown in panel (c) of Figure 1, when σ_V is large, ignoring the default risk of counter party and directly using the Black-Scholes formula to assess an option which is actually vulnerable will cause significant overpricing. This is due to the great default risk of counter party when its volatility σ_V is large or initial firm value $V(0)$ is too small. In contrast, when $V(0)$ is large or σ_V too small, the value of a vulnerable option tends to that of a non-vulnerable one. That is, an option holder who buy the option from a small firm, in contrast to a big firm, is more vulnerable and hence needs more risk premium, which is consistent with the intuition. In case of barrier options, all the above statements still hold and can be observed in panel (a), (b), (c) of Figure 2, which compare the values of vulnerable and non-vulnerable barrier options based on the settings $S(0)=40, K=40, V(0)=100, T=3, D=90, B=35, \sigma_V=0.2, \sigma_S=0.2, \rho=0, r=0.05, \gamma=0, \alpha=0.25$.

6 Conclusion

We extend the works of Klein (1996) to obtain an analytical pricing formula for vulnerable options, releasing the constrain of credit risk model used to measure the default risk of counter-party. A more flexible model, the first passage model, is used in place of the original Merton's model. Analytical pricing formulae for vulnerable barrier options are also derived by similar methodology. Furthermore, these pricing scheme can also be used to derive the pricing formulae for structured notes. As examples, the derivations of the values of reverse exchangeable bond and barrier reverse convertibles are given. The pricing results compared to numerical method developed by Liu (2020) are also given to confirm the validity of our formulae.

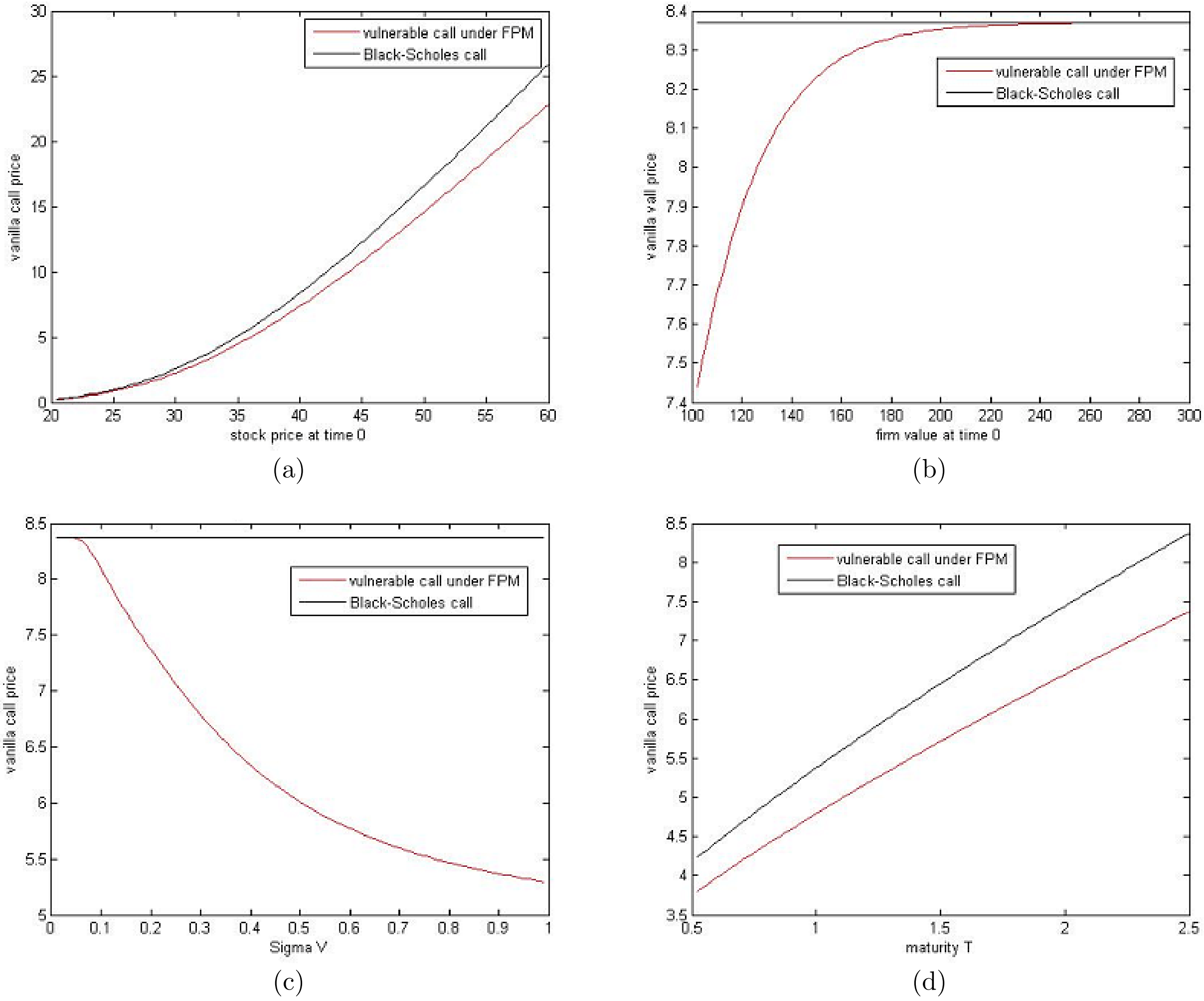
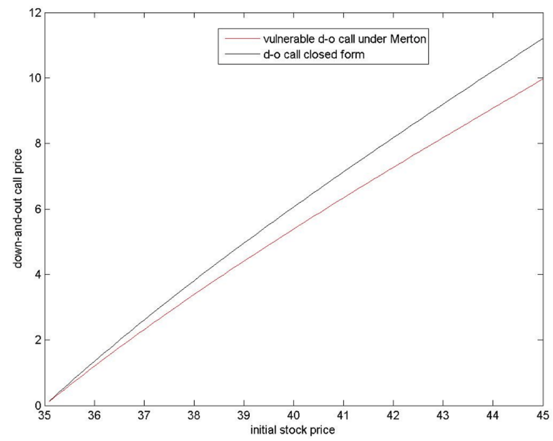
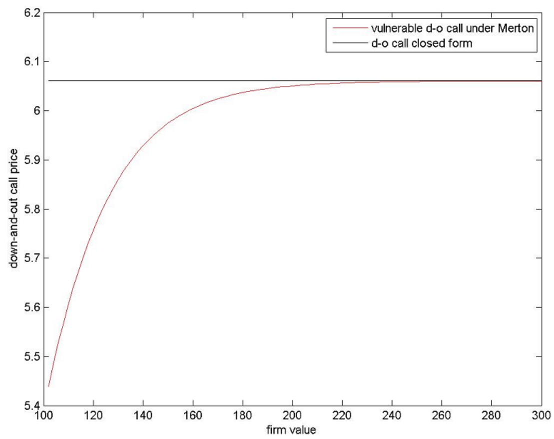


Figure 1: Comparison of the Value of the Vulnerable and Non-vulnerable Vanilla Options

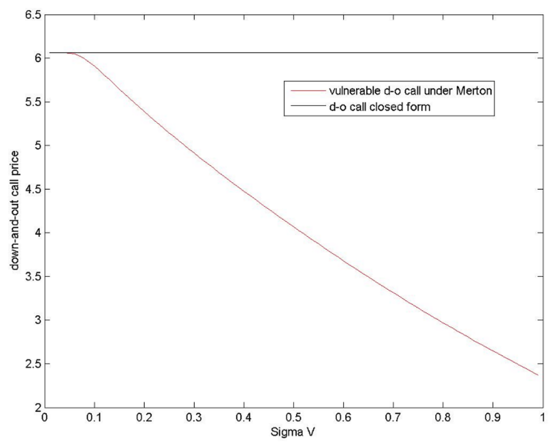
The value of the vulnerable and non-vulnerable vanilla options based on the setting $S(0) = 40, K = 40, V(0) = 100, T = 3, D = 90, L = 70, \sigma_V = 0.2, \rho = 0, r = 0.05, \gamma = 0.06, \delta = 0.6$, unless otherwise noted. The option values under different $S(0)$, $V(0)$, σ_V, T are shown in panel (a), (b), (c), and (d) respectively. The black line denotes the vulnerable vanilla call value evaluated by Eq. (12), and the red line, the non-vulnerable vanilla call value evaluated by Black-Scholes fomrula.



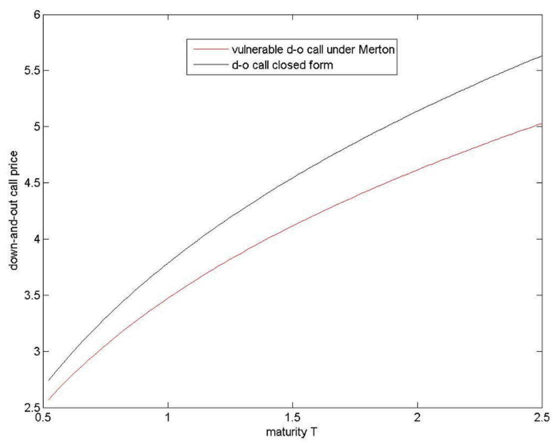
(a)



(b)



(c)



(d)

Figure 2: Comparison of the Value of the Vulnerable and Non-vulnerable Barrier Options

The value of the vulnerable and non-vulnerable vanilla options based on the setting $S(0) = 40, K = 40, V(0) = 100, T = 3, D = 90, B = 35, \sigma_V = 0.2, \sigma_S = 0.2, \rho = 0, r = 0.05, \gamma = 0, \alpha = 0.25$, unless otherwise noted. The option values under different $S(0)$, $V(0)$, σ_V, T are shown in panel (a), (b), (c), and (d) respectively. The black line denotes the vulnerable down-and-out call value evaluated by Eq. (12), and the red line, the non-vulnerable down-and-out call value evaluated by Rubinstein and Reiner (1991).

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A The Joint Probability Density Function of $M_1(T), B_1(T), B_2(T)$

Assume $W_1(t)$, $W_2(t)$ are independent standard Brownian motion and set

$$\begin{aligned} B_1(t) &\equiv \alpha t + W_1(t), \\ B_2(t) &\equiv \beta t + \rho W_1(t) + \sqrt{1-\rho^2} W_2(t). \end{aligned}$$

Notice that $B_1(t), B_2(t)$ are correlated Brownian motions with drift terms, and satisfy $dB_1(t)dB_2(t) = \rho dt$. Define the minimum value process $M_1(t)$ as

$$M_1(t) \equiv \min_{s\in[0,t]} B_1(s).$$

First we need the joint density of $M_1(T), B_1(T)$ given in the following lemma:

Lemma A.1 *The joint probability density function of $M_1(T), B_1(T)$ is given by*

$$f_{M_1(T),B_1(T)}(m,b) = \begin{cases} \frac{2(b-2m)}{T\sqrt{2\pi T}} e^{\alpha b - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-b)^2} & \textit{if } m \le b^- \\ 0 & \textit{otherwise} \end{cases}, \tag{29}$$

where b^- stands for $\min(b, 0)$.

Proof. Let $Z(t)$ be a standard Brownian motion, $\hat{Z}(t) \equiv \lambda t + Z(t)$ be a Brownian motion with drift term, and $\hat{M}(t) \equiv \max_{s\in[0,t]} \hat{Z}(s)$ be its maximum value process. As mentioned in Shreve (2004), the joint density function of $(\hat{M}(T), \hat{Z}(T))$ is given by

$$f_{\hat{M}(T),\hat{Z}(T)}(\hat{m},\hat{z}) \begin{cases} \frac{2(2\hat{m}-\hat{z})}{T\sqrt{2\pi T}} e^{\lambda \hat{z} - \frac{1}{2}\lambda^2 T - \frac{1}{2T}(2\hat{m}-\hat{z})^2} & \text{if } \hat{m} \ge \hat{z}^+ \\ 0 & \text{otherwise} \end{cases}.$$

Note that the density of $(-\hat{M}(T), -\hat{Z}(T))$ is given by $f_{-\hat{M}(T),-\hat{Z}(T)}(m,z) = f_{\hat{M}(T),\hat{Z}(T)}(-m,-z)$. Since both $-Z(t)$ and $W_1(t)$ are standard Brownian motions, $\big(\min_{s\in[0,T]}(-\lambda s + W_1(s))\ ,\ -\lambda T + W_1(T)\big)$ has the same density as $(-\hat{M}(T), -\hat{Z}(T)) = \big(\min_{s\in[0,T]}(-\lambda s - Z(s))\ ,\ -\lambda T - Z(T)\big)$. The joint density of $(M_1(T), B_1(T))$ is obtained by substituting $\alpha = -\lambda$ into Eq. (29).

□

To derive the joint probability density function of $M_1(T), B_1(T), B_2(T)$, rewrite the following probability as:

$$\begin{aligned} & P\{M_1(T) < m_1, B_1(T) < b_1, B_2(T) < b_2\} \\ = \; & P\left\{ M_1(T) < m_1, B_1(T) < b_1, W_2(T) < \frac{b_2 - \beta T + \rho\alpha T - (\rho\alpha T + \rho W_1(T))}{\sqrt{1-\rho^2}} \right\} \\ = \; & P\left\{ M_1(T) < m_1, B_1(T) < b_1, W_2(T) < \frac{b_2 + (\rho\alpha - \beta)T - \rho B_1(T)}{\sqrt{1-\rho^2}} \right\} \\ = \; & \int_{-\infty}^{m_1} \int_{-\infty}^{b_1} \int_{-\infty}^{\frac{b_2 + (\rho\alpha-\beta)T - \rho b}{\sqrt{1-\rho^2}}} f_{M_1(T),B_1(T)}(m,b) f_{W_2(T)}(w) dw db dm, \end{aligned}$$

where $f_{M_1(T),B_1(T)}$, $f_{W_2(T)}$ denote the density function of $(M_1(T), B_1(T))$ and $W_2(T)$, respectively. Differentiate the above integral to obtain the joint density function of $M_1(T), B_1(T), B_2(T)$ as follows:

$$\begin{aligned} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2) &= \frac{\partial}{\partial m_1 \partial b_1 \partial b_2} \int_{-\infty}^{m_1} \int_{-\infty}^{b_1} \int_{-\infty}^{\frac{b_2 + (\rho\alpha-\beta)T - \rho b_1}{\sqrt{1-\rho^2}}} f_{M_1(T),B_1(T)}(m,b) f_{W_2(T)}(w) dw db dm \\ &= \frac{1}{\sqrt{1-\rho^2}} f_{M_1(T),B_1(T)}(m_1,b_1) f_{W_2(T)}\left(\frac{b_2 + (\rho\alpha - \beta)T - \rho b_1}{\sqrt{1-\rho^2}} \right). \end{aligned}$$

Applying Eq. (29), we obtain

$$\begin{aligned} & f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2) \\ = \; & \begin{cases} \frac{1}{\sqrt{1-\rho^2}} \frac{2(b_1-2m_1)}{T\sqrt{2\pi T}} e^{\alpha b_1 - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(b_1 - 2m_1)^2} \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T(1-\rho^2)}(b_2 + (\rho\alpha-\beta)T - \rho b_1)^2} & \text{if } m_1 \le b_1^- \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{30}$$

B Proof of Theorem 3.1

Note that

$$\begin{aligned}
&\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty} e^{pb_1+qb_2} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2\\
=\ &\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{b_1^-} e^{pb_1+qb_2}\frac{1}{\sqrt{1-\rho^2}}\frac{2(b_1-2m_1)}{T\sqrt{2\pi T}}e^{\alpha b_1-\frac{1}{2}\alpha^2T-\frac{1}{2T}(b_1-2m_1)^2}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T(1-\rho^2)}(b_2+(\rho\alpha-\beta)T-\rho b_1)^2}dm_1db_1db_2\\
=\ &\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty} e^{pb_1+qb_2}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T(1-\rho^2)}(b_2+(\rho\alpha-\beta)T-\rho b_1)^2}\left(\int_{\bar{m}_1}^{b_1^-}\frac{2(b_1-2m_1)}{T\sqrt{2\pi T}}e^{\alpha b_1-\frac{1}{2}\alpha^2T-\frac{1}{2T}(b_1-2m_1)^2}dm_1\right)db_1db_2,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&\int_{\bar{m}_1}^{b_1^-}\frac{2(b_1-2m_1)}{T\sqrt{2\pi T}}e^{\alpha b_1-\frac{1}{2}\alpha^2T-\frac{1}{2T}(b_1-2m_1)^2}dm_1\\
=\ &\left[\frac{1}{\sqrt{2\pi T}}e^{\alpha b_1-\frac{1}{2}\alpha^2T-\frac{1}{2T}(b_1-2m_1)^2}\right]_{m_1=\bar{m}_1}^{m_1=b_1^-}=\frac{1}{\sqrt{2\pi T}}e^{\alpha b_1-\frac{1}{2}\alpha^2T-\frac{b_1^2}{2T}}\left(1-e^{\frac{2\bar{m}_1(b_1-\bar{m}_1)}{T}}\right).
\end{aligned}$$

Thus, we obtain from Eq. (31) that

$$\begin{aligned}
&\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty} e^{pb_1+qb_2} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2\\
=\ &\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty} e^{pb_1+qb_2}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{2\pi T}e^{-\frac{1}{2T(1-\rho^2)}(b_2+(\rho\alpha-\beta)T-\rho b_1)^2+\alpha b_1-\frac{1}{2}\alpha^2T-\frac{b_1^2}{2T}}\left(1-e^{\frac{2\bar{m}_1(b_1-\bar{m}_1)}{T}}\right)db_1db_2\\
=\ &\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty} e^{pb_1+qb_2}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{2\pi T}e^{-\frac{1}{2T(1-\rho^2)}(b_2+(\rho\alpha-\beta)T-\rho b_1)^2+\alpha b_1-\frac{1}{2}\alpha^2T-\frac{b_1^2}{2T}}db_1db_2 &(32)\\
&-e^{-\frac{2\bar{m}_1^2}{T}}\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty} e^{\left(p+\frac{2\bar{m}_1}{T}\right)b_1+qb_2}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{2\pi T}e^{-\frac{1}{2T(1-\rho^2)}(b_2+(\rho\alpha-\beta)T-\rho b_1)^2+\alpha b_1-\frac{1}{2}\alpha^2T-\frac{b_1^2}{2T}}db_1db_2. &(33)
\end{aligned}$$

To simplify the above integral, the following lemma is needed:

Lemma B.1

$$\begin{aligned}
&\int_{\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty} e^{C_1b_1+C_2b_2}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{2\pi T}e^{-\frac{1}{2T(1-\rho^2)}(b_2+(\rho\alpha-\beta)T-\rho b_1)^2+\alpha b_1-\frac{1}{2}\alpha^2T-\frac{b_1^2}{2T}}db_1db_2\\
=\ &e^{T(\frac{C_1^2}{2}+\rho C_1C_2+\frac{C_2^2}{2}+\alpha C_1+\beta C_2)}N_2\left(-\frac{\bar{b}_1}{\sqrt{T}}+\sqrt{T}(\alpha+C_1+\rho C_2),-\frac{\bar{b}_2}{\sqrt{T}}+\sqrt{T}(\beta+\rho C_1+C_2),\rho\right).
\end{aligned}$$

Proof. Substituting

$$\begin{cases} b_1=-\sqrt{T}x+T(\alpha+C_1+\rho C_2)\\ b_2=-\sqrt{T}y+T(\beta+\rho C_1+C_2)\end{cases},$$

the left hand side integral can be rewritten as

$$\begin{aligned}
&e^{T(\frac{C_1^2}{2}+\rho C_1C_2+\frac{C_2^2}{2}+\alpha C_1+\beta C_2)}\int_{-\frac{\bar{b}_2}{\sqrt{T}}+\sqrt{T}(\beta+\rho C_1+C_2)}^{-\infty}\int_{-\frac{\bar{b}_1}{\sqrt{T}}+\sqrt{T}(\alpha+C_1+\rho C_2)}^{-\infty}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{2\pi}e^{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)}dxdy\\
=\ &e^{T(\frac{C_1^2}{2}+\rho C_1C_2+\frac{C_2^2}{2}+\alpha C_1+\beta C_2)}\int_{-\infty}^{-\frac{\bar{b}_2}{\sqrt{T}}+\sqrt{T}(\beta+\rho C_1+C_2)}\int_{-\infty}^{-\frac{\bar{b}_1}{\sqrt{T}}+\sqrt{T}(\alpha+C_1+\rho C_2)}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{2\pi}e^{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)}dxdy,
\end{aligned}$$

which, by its very definition, equals the right hand side. □

Obviously, Eq. (32) can be rewritten as

$$e^{T\left(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q\right)}N_2\left(-\frac{\bar{b}_1}{\sqrt{T}}+\sqrt{T}(\alpha+p+\rho q),-\frac{\bar{b}_2}{\sqrt{T}}+\sqrt{T}(\beta+\rho p+q),\rho\right)$$

immediately applying the lemma B.1. Also, substituting $C_1=p+\frac{2\bar{m}_1}{T}$, $C_2=q$ into this lemma yields the simplified form of Eq. (33) as

$$-e^{T\left(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q\right)+2\bar{m}_1(p+\rho q)+2\alpha\bar{m}_1}N_2\left(-\frac{\bar{b}_1-2\bar{m}_1}{\sqrt{T}}+\sqrt{T}\left(\alpha+p+\rho q\right),-\frac{\bar{b}_2-2\rho\bar{m}_1}{\sqrt{T}}+\sqrt{T}\left(\beta+\rho p+q\right),\rho\right).$$

Summing up, we obtain Theorem 3.1.

C Bivariate normal cdf reduces to univariate normal

$$
\begin{aligned}
\lim_{a\to\infty} N_2(a,b,\rho) &= \int_{-\infty}^{b}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)}dxdy \\
&= \int_{-\infty}^{b}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{(x-\rho y)^2+y^2(1-\rho^2)}{2(1-\rho^2)}}dxdy \\
&= \int_{-\infty}^{b}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)^2}\frac{dx}{\sqrt{1-\rho^2}}\right)dy \\
&= \int_{-\infty}^{b}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy = N_1(b).
\end{aligned}
$$

D Proof of Theorem 4.1

We start by substituting $-b_3 = b_2$ into the left-hand side of Eq. (24) to obtain

$$
\begin{aligned}
&\int_{-\infty}^{\bar{b}_2}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty} e^{pb_1+qb_2} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,b_2)dm_1db_1db_2 \\
=\;& \int_{-\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty} e^{pb_1-qb_3} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,-b_3)dm_1db_1db_3,
\end{aligned}
\tag{34}
$$

where, by Eq. (30), $f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,-b_3)$ is known as

$$
\begin{cases}
\frac{1}{\sqrt{1-\rho^2}}\frac{2(b_1-2m_1)}{T\sqrt{2\pi T}}e^{\alpha b_1-\frac{1}{2}\alpha^2T-\frac{1}{2T}(b_1-2m_1)^2}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T(1-\rho^2)}(-b_3+(\rho\alpha-\beta)T-\rho b_1)^2} & \text{if } m_1 \le b_1^- \\
0 & \text{otherwise}
\end{cases}.
\tag{35}
$$

Let $B_3(t) \equiv -\beta t - \rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$, which satisfy $dB_1(t)dB_3(t) = -\rho dt$. Again by Eq. (30) we have

$$
\begin{aligned}
&f_{M_1(T),B_1(T),B_3(T)}(m_1,b_1,b_3) \\
=\;& \begin{cases}
\frac{1}{\sqrt{1-\rho^2}}\frac{2(b_1-2m_1)}{T\sqrt{2\pi T}}e^{\alpha b_1-\frac{1}{2}\alpha^2T-\frac{1}{2T}(b_1-2m_1)^2}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T(1-\rho^2)}(b_3+(-\rho\alpha+\beta)T+\rho b_1)^2} & \text{if } m_1 \le b_1^- \\
0 & \text{otherwise}
\end{cases},
\end{aligned}
$$

which is exactly the same as Eq. (35). Thus, Eq. (34) can be replaced by

$$
\begin{aligned}
&\int_{-\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty} e^{pb_1-qb_3} f_{M_1(T),B_1(T),B_2(T)}(m_1,b_1,-b_3)dm_1db_1db_3, \\
=\;& \int_{-\bar{b}_2}^{\infty}\int_{\bar{b}_1}^{\infty}\int_{\bar{m}_1}^{\infty} e^{pb_1-qb_3} f_{M_1(T),B_1(T),B_3(T)}(m_1,b_1,b_3)dm_1db_1db_3.
\end{aligned}
$$

Now it can be simplified by Theorem 3.1 immediately as

$$
\begin{aligned}
&e^{T(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q)}N_2\left(-\frac{\bar{b}_1}{\sqrt{T}}+\sqrt{T}(\alpha+p+\rho q), \frac{\bar{b}_2}{\sqrt{T}}-\sqrt{T}(\beta+\rho p+q), -\rho\right) \\
&-e^{T\left(\frac{p^2}{2}+\rho pq+\frac{q^2}{2}+\alpha p+\beta q\right)+2\bar{m}_1(p+\rho q)+2\alpha\bar{m}_1}N_2\left(-\frac{\bar{b}_1-2\bar{m}_1}{\sqrt{T}}+\sqrt{T}\left(\alpha+p+\rho q\right), \frac{\bar{b}_2-2\rho\bar{m}_1}{\sqrt{T}}-\sqrt{T}\left(\beta+\rho p+q\right), -\rho\right),
\end{aligned}
$$

which is just the right-hand side of Eq. (24) and the proof is complete.

E Barrier option pricing formulae

As mentioned before, by introducing in-out parity, we only need C_{DO}, C_{UO}, P_{UI} and P_{DI} to evaluate all 8 types of vulnerable barrier options. In these section we list the analytical pricing formulae for all of them except C_{DO}, which is given in Eq. (25).

Under Merton's credit risk model with default boundary and barrier settings the same as section 4, C_{UO}, P_{UI}, P_{DI} are given by

$$
\begin{aligned}
C_{UO} = \quad & S_0 \left[N_2\left(f_1 + \frac{b}{\sqrt{T}}, -e_1, -\rho \right) - N_2\left(f_1 + \frac{k'}{\sqrt{T}}, -e_1, -\rho \right) \right] \\
- \quad & S_0 e^{2b\sigma_S - 2b\alpha_S} \left[N_2\left(f_2 + \frac{b}{\sqrt{T}}, -e_2, -\rho \right) - N_2\left(f_2 + \frac{k'}{\sqrt{T}}, -e_2, -\rho \right) \right] \\
- \quad & K e^{-rT} \left[N_2\left(f_3 + \frac{b}{\sqrt{T}}, -e_3, -\rho \right) - N_2\left(f_3 + \frac{k'}{\sqrt{T}}, -e_3, -\rho \right) \right] \\
+ \quad & K e^{-rT - 2b\alpha_S} \left[N_2\left(f_4 + \frac{b}{\sqrt{T}}, -e_4, -\rho \right) - N_2\left(f_4 + \frac{k'}{\sqrt{T}}, -e_4, -\rho \right) \right] \\
+ \quad & \left(\frac{1-\alpha}{D} \right) V_0 S_0 e^{\bar{\gamma}T - 2\sigma_v \rho \alpha_S T + \frac{1}{2}\sigma_S^2 T + \rho\sigma_S\sigma_V T - \sigma_S \alpha_S T} \left[N_2\left(f_5 + \frac{b}{\sqrt{T}}, e_5, \rho \right) - N_2\left(f_5 + \frac{k'}{\sqrt{T}}, e_5, \rho \right) \right] \\
- \quad & \left(\frac{1-\alpha}{D} \right) V_0 S_0 e^{\bar{\gamma}T - 2\sigma_v \rho \alpha_S T + \frac{1}{2}\sigma_S^2 T + \rho\sigma_S\sigma_V T - \sigma_S \alpha_S T - 2\alpha_S b + 2b\rho\sigma_V + 2b\sigma_S} \\
& \cdot \left[N_2\left(f_6 + \frac{b}{\sqrt{T}}, e_6, \rho \right) - N_2\left(f_6 + \frac{k'}{\sqrt{T}}, e_6, \rho \right) \right] \\
- \quad & \left(\frac{1-\alpha}{D} \right) K V_0 e^{-2\sigma_V \rho \alpha_S T} \left[N_2\left(f_7 + \frac{b}{\sqrt{T}}, e_7, \rho \right) - N_2\left(f_7 + \frac{k'}{\sqrt{T}}, e_7, \rho \right) \right] \\
+ \quad & \left(\frac{1-\alpha}{D} \right) K V_0 e^{-2\sigma_V \rho \alpha_S T - 2\alpha_S b + 2b\rho\sigma_V} \left[N_2\left(f_8 + \frac{b}{\sqrt{T}}, e_8, \rho \right) - N_2\left(f_8 + \frac{k'}{\sqrt{T}}, e_8, \rho \right) \right], \\
P_{UI} = \quad & K e^{-rT - 2b\alpha_S} N_2\left(f_4 + \frac{k'}{\sqrt{T}}, -e_4, -\rho \right) \\
- \quad & S_0 e^{\bar{\gamma}T - rT + \frac{1}{2}\sigma_S^2 T - \alpha_S \sigma_S T + 2b\sigma_S - 2\alpha_S b} N_2\left(f_2 + \frac{k'}{\sqrt{T}}, -e_2, -\rho \right) \\
+ \quad & \left(\frac{1-\alpha}{D} \right) K V_0 e^{-2\sigma_V \rho \alpha_S T - 2\alpha_S b + 2b\rho\sigma_V} N_2\left(f_8 + \frac{k'}{\sqrt{T}}, e_8, \rho \right) \\
- \quad & \left(\frac{1-\alpha}{D} \right) V_0 S_0 e^{\bar{\gamma}T - 2\sigma_V \rho \alpha_S T + \frac{1}{2}\sigma_S^2 T + \rho\sigma_S\sigma_V T - \sigma_S \alpha_S T - 2\alpha_S b + 2b\rho\sigma_V + 2b\sigma_S} N_2\left(f_6 + \frac{k'}{\sqrt{T}}, e_6, \rho \right), \\
P_{DI} = \quad & K e^{-rT} N_2\left(-h_3 + \frac{b}{\sqrt{T}}, -g_3, -\rho \right) \\
+ \quad & K e^{-rT + 2b\alpha_S} \left[N_2\left(h_4 - \frac{b}{\sqrt{T}}, -g_4, \rho \right) - N_2\left(h_4 - \frac{k'}{\sqrt{T}}, -g_4, \rho \right) \right] \\
- \quad & S_0 N_2\left(-h_1 + \frac{b}{\sqrt{T}}, -g_1, -\rho \right) \\
- \quad & S_0 e^{2b\sigma_S + 2b\alpha_S b} \left[N_2\left(h_2 - \frac{b}{\sqrt{T}}, -g_2, \rho \right) - N_2\left(h_2 - \frac{k'}{\sqrt{T}}, -g_2, \rho \right) \right] \\
+ \quad & \left(\frac{1-\alpha}{D} \right) K V_0 N_2\left(-h_7 + \frac{b}{\sqrt{T}}, g_7, \rho \right) \\
+ \quad & \left(\frac{1-\alpha}{D} \right) K V_0 e^{2\alpha_S b + 2b\rho\sigma_V} \left[N_2\left(h_8 - \frac{b}{\sqrt{T}}, g_8, -\rho \right) - N_2\left(h_8 - \frac{k'}{\sqrt{T}}, g_8, -\rho \right) \right] \\
- \quad & \left(\frac{1-\alpha}{D} \right) V_0 S_0 e^{rT + \rho\sigma_S\sigma_V T} N_2\left(-h_5 + \frac{b}{\sqrt{T}}, g_5, \rho \right) \\
- \quad & \left(\frac{1-\alpha}{D} \right) V_0 S_0 e^{rT + \rho\sigma_S\sigma_V T + 2\alpha_S b + 2b\rho\sigma_V + 2b\sigma_S} \left[N_2\left(h_6 - \frac{b}{\sqrt{T}}, g_6, -\rho \right) - N_2\left(h_6 - \frac{k'}{\sqrt{T}}, g_6, -\rho \right) \right],
\end{aligned}
$$

where

$$
\alpha_S \equiv \frac{1}{\sigma_S}(r - \bar{\gamma} - \sigma_S^2/2),
$$

$$
\begin{aligned}
e_1 &\equiv e_3 - \sigma_S \rho \sqrt{T} & f_1 &\equiv -\sigma_S \sqrt{T} + \alpha_S \sqrt{T} \\
e_3 &\equiv \frac{\ln \frac{D}{V_0} - \left(r - \frac{1}{2}\sigma_V^2\right)}{\sigma_V \sqrt{T}} + 2\rho\alpha_S \sqrt{T} & f_3 &\equiv \alpha_S \sqrt{T} \\
e_5 &\equiv e_3 - \sigma_S \rho \sqrt{T} - \sigma_V \sqrt{T} & f_5 &\equiv -\sigma_S \sqrt{T} - \sigma_V \rho \sqrt{T} + \alpha_S \sqrt{T} \\
e_7 &\equiv e_3 - \sigma_V \sqrt{T} & f_7 &\equiv -\sigma_V \rho \sqrt{T} + \alpha_S \sqrt{T}
\end{aligned}
$$

$$
\begin{aligned}
g_1 &\equiv g_3 - \sigma_S \rho \sqrt{T} & h_1 &\equiv \sigma_S \sqrt{T} + \alpha_S \sqrt{T} \\
g_3 &\equiv \frac{\ln \frac{D}{V_0} - \left(r - \frac{1}{2}\sigma_V^2\right)}{\sigma_V \sqrt{T}} & h_3 &\equiv \alpha_S \sqrt{T} \\
g_5 &\equiv g_3 - \sigma_V \sqrt{T} - \sigma_S \rho \sqrt{T} & h_5 &\equiv \sigma_S \sqrt{T} + \sigma_V \rho \sqrt{T} + \alpha_S \sqrt{T} \\
g_7 &\equiv g_3 - \sigma_V \sqrt{T} & h_7 &\equiv \sigma_V \rho \sqrt{T} + \alpha_S \sqrt{T}
\end{aligned}
$$

$$
\begin{aligned}
e_i &\equiv e_{i-1} - \tfrac{2b\rho}{\sqrt{T}}, & f_i &\equiv f_{i-1} - \tfrac{2b}{\sqrt{T}}, \\
g_i &\equiv g_{i-1} - \tfrac{2b\rho}{\sqrt{T}}, & h_i &\equiv h_{i-1} + \tfrac{2b}{\sqrt{T}} \quad \forall i = 2, 4, 6, 8.
\end{aligned}
$$